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ON SHAPE OF PRODUCT SPACES

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It is known that if X is a compactum and Y is metrizable $\mathrm{Sh}_s(X \times Y)$ is not generally determined by $\mathrm{Sh}_s(X)$ and $\mathrm{Sh}_s(Y)$, where $\mathrm{Sh}_s(Z)$ is the strong shape of Z in the sense of Borsuk. In this paper it is proved that $\mathrm{Sh}(X \times Y)$ is uniquely determined by $\mathrm{Sh}(X)$ and $\mathrm{Sh}(Y)$, where $\mathrm{Sh}(Z)$ is the shape of Z in the sense of Fox. If X is an FANR and Y is an MANR, then $X \times Y$ is an MANR.

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shape	product space
fundamental dimension	mutational neighborhood retract

Dedicated to Professor Karol Borsuk for his 70th birthday

1. Introduction

The notion of shape has been introduced by Borsuk [1] for compacta. This notion was generalized to arbitrary metrizable spaces by Borsuk [2] and Fox [6]. These shapes are equivalent for compacta but not generally for non compact spaces. For a metrizable space X we denote by $\mathrm{Sh}_s(X)$ the strong shape of X in the sense of Borsuk and by $\mathrm{Sh}(X)$ the shape of X in the sense of Fox.

In this paper we shall prove that if X is a compactum and Y is metrizable then $\mathrm{Sh}(X \times Y)$ is uniquely determined by $\mathrm{Sh}(X)$ and $\mathrm{Sh}(Y)$. It is known that $\mathrm{Sh}_s(X \times Y)$ is not generally determined by $\mathrm{Sh}_s(X)$ and $\mathrm{Sh}_s(Y)$. As an application we obtain a product theorem for fundamental dimension: If X is a compactum and Y is metrizable then $\mathrm{Fd}(X \times Y) \leq \mathrm{Fd}(X) + \mathrm{Fd}(Y)$, where $\mathrm{Fd}(X)$ is the fundamental dimension of X in the sense of Borsuk [3]. Finally we shall show that if X is an FAR (resp. FANR) and Y is an MAR (resp. MANR) then $X \times Y$ is an MAR (resp. MANR). Here we mean by an FAR (FANR) a fundamental absolute (neighborhood) retract in the sense of Borsuk [3, Ch. VIII] and by an MAR(MANR) a mutational absolute (neighborhood) retract in the sense of Godlewski [7].

Throughout the paper all spaces are metrizable and maps are continuous. AR and ANR mean those for metric spaces.

2. Basic notions and lemmas

Let X be a compactum (= a compact metric space). According to Mardešić and Segal [14] an inverse sequence of compact ANR's $\boldsymbol{X} = \{X_k, \pi_k^{k+1}, k \in J\}$ (J = the set of positive integers) such that $\varprojlim \boldsymbol{X} = X$ is said to be an ANR-*sequence associated with* X. As in [11], for a given ANR-sequence $\boldsymbol{X} = \{X_k, \pi_k^{k+1}\}$ associated with X we construct an AR $N(\boldsymbol{X})$ containing X as follows. For each $k \in J$, $M_k(\boldsymbol{X})$ denotes the mapping cylinder obtained by X_k, X_{k+1} and $\pi_k^{k+1}: X_{k+1} \to X_k$, that is, $M_k(\boldsymbol{X})$ is obtained by identifying points $(x, 0) \in X_{k+1} \times \{0\}$ and $\pi_k^{k+1}(x) \in X_k$ for $x \in X_{k+1}$ in a topological sum $X_{k+1} \times I \cup X_k$, where $I = [0, 1]$. Let $M_0(\boldsymbol{X})$ be a cone over X_1. Consider a topological sum $T = \bigcup_{k=0}^{\infty} M_k(\boldsymbol{X})$. For each $k \in J$, by identifying point of $X_k \times \{1\}$ in $M_{k-1}(\boldsymbol{X})$ and the corresponding point of X_k in $M_k(\boldsymbol{X})$ we obtain from T a locally compact metrizable space $T(\boldsymbol{X})$ which is said to be an *infinite telescope* associated with $\boldsymbol{X}$. Set $N(\boldsymbol{X}) = T(\boldsymbol{X}) \cup X$. Give $N(\boldsymbol{X})$ the following topology. $T(\boldsymbol{X})$ is open in $N(\boldsymbol{X})$ with its proper topology. Let $x \in X$. For each k, let V be an open neighborhood of $\pi_k(x)$ in X_k, where π_k is the projection of X into X_k. For $m > k$, consider an open set $(\pi_k^m)^{-1} V \times (0, 1]$ of $M_{m-1}(\boldsymbol{X})$, where $(0, 1] = \{t : 0 < t \leq 1\}$ and $\pi_k^m = \pi_k^{k+1} \cdots \pi_{m-1}^m$. The collection of the sets of the form

$$\pi_k^{-1}(V) \cup \bigcup_{m=k+1}^{\infty} (\pi_k^m)^{-1} V \times (0, 1],$$

where V ranges over open neighborhoods of $\pi_k(x)$ in X_k for $k \in J$, forms a neighborhood basis of x in $N(\boldsymbol{X})$. By [11, Theorem 1] $N(\boldsymbol{X})$ is an AR and contains X and $M_k(\boldsymbol{X})$, $k \in J$, as closed sets. $N(\boldsymbol{X})$ is said to be an AR *associated with* $\boldsymbol{X}$.

Let R_0 be the subspace of the real line consisting of non-negative real numbers. For each $r \in R_0$ we define a retraction φ_r of $N(\boldsymbol{X})$ as follows. Let $r = n + s$, where n is the largest integer not greater than r and $0 \leq s < 1$. Set

$$\varphi_r(x) = (\pi_{n+1}(x), s), \qquad x \in X,$$

$$\varphi_r(x, t) \begin{cases} = (\pi_{n+1}^m(x, t), s), & (x, t) \in M_m(\boldsymbol{X}), \quad m = n+2, n+3, \ldots, \\ = (x, s), & (x, t) \in M_{n+1}(\boldsymbol{X}), \quad t \leq s, \\ = (x, t), & (x, t) \in M_{n+1}(\boldsymbol{X}), \quad t \geq s, \\ = (x, t), & (x, t) \in M_m(\boldsymbol{X}), \quad m = 0, 1, \ldots, n. \end{cases}$$

For each $r \in R_0$, we put

$$N_r(\boldsymbol{X}) = \bigcup_{m=0}^{n-1} M_m(\boldsymbol{X}) \cup X_{n+1} \times (0, s],$$

where $r = n + s$, $0 \leq s < 1$. It is easy to know that $N_r(\boldsymbol{X})$ is a strong deformation retract of $N(\boldsymbol{X})$ and φ_r is a retraction on it.

Lemma 2.1. *A function* $\varphi : R_0 \times N(\boldsymbol{X}) \to N(\boldsymbol{X})$ *defined by* $\varphi(r, x) = \varphi_r(x)$ *for* $r \in R_0$ *and* $x \in N(\boldsymbol{X})$ *is continuous.*

The proof is obvious.

Let M be a metrizable space. We denote by $C(M, R_0)$ the set of all (continuous) maps of M into R_0. For each map $f \in C(M, R_0)$ we define a subset M_f of the product space $N(X) \times M$ which is said to be an *f-set* and a function $\alpha_f : N(X) \times M \to N(X) \times M$ as follows.

$$M_f = \{(\varphi_{f(y)}(x), y) : (x, y) \in N(X) \times M\}, \tag{2.1}$$

$$\alpha_f(x, y) = (\varphi_{f(y)}, y), \quad (x, y) \in N(X) \times M. \tag{2.2}$$

From the definition

$$\alpha_f \mid M_f = \text{the identity}. \tag{2.3}$$

Lemma 2.2. (1) *α_f is continuous and the f-set $\alpha_f(N(X) \times M) = M_f$ is a closed set disjoint from $X \times M$.*

(2) *For any closed set B of $N(X) \times M$ disjoint from $X \times M$, there is a map $f \in C(M, R_0)$ such that $B \subset M_f$.*

(3) *For any $f, g \in C(M, R_0)$, there exists a homotopy $H : N(X) \times M \times I \to N(X) \times M$ such that $H \mid N(X) \times M \times \{0\} = \alpha_f$, $H \mid N(X) \times M \times \{1\} = \alpha_g$ and $H(x, y, t) = (x, y)$ for $(x, y) \in M_f \cap M_g$ and $t \in I$.*

Proof. (1) Consider the functions $\alpha : N(X) \times M \to R_0 \times N(X) \times M$ and $\beta : R_0 \times N(X) \times M \to N(X) \times M$ is defined by

$$\alpha(x, y) = (f(y), x, y), \qquad (x, y) \in N(X) \times M,$$

$$\beta(r, x, y) = (\varphi_r(x), y), \qquad (r, x, y) \in R_0 \times N(X) \times M.$$

Obviously α is continuous. Also, by Lemma 2.1, β is continuous. Since $\alpha_f = \beta\alpha$, α_f is continuous. The closedness of M_f is obvious.

(2) Since X is compact, for each point $y \in M$ there exists a neighborhood V_y in M and $r_y \in R_0$ such that $N_{r_y}(X) \times V_y \supset B \cap (N(X) \times V_y)$. From this fact and the paracompactness of M a required map f is obtained easily.

(3) Define $\psi_0, \psi_1 \in C(M, R_0)$ by $\psi_0(y) = \min\{f(y), g(y)\}$ and $\psi_1(y) = \max\{f(y), g(y)\}$ for $y \in M$. Then a required homotopy H is given as follows.

$$H(x, y, t) = \alpha_{(1-t)\psi_0 + t\psi_1}(x, y) \quad \text{for } (x, y) \in N(X) \times M \text{ and } t \in I.$$

Let Y be a closed subset of a space M. By $\mathrm{U}(Y, M)$ we denote the complete system of neighborhoods of Y in M. For a neighborhood $V \in \mathrm{U}(Y, M)$ and a map $f \in C(M, R_0)$ the subset $V_f = N(X) \times V - M_f$ of $N(X) \times M$ is a neighborhood of $X \times Y$ in $N(X) \times M$, that is, $V_f \in \mathrm{U}(X \times Y, N(X) \times M)$. V_f is said to be a *basic neighborhood.* If $V \subset V'$, $V, V' \in \mathrm{U}(Y, M)$ and $f \geq f'$, $f, f' \in C(M, R_0)$ (that is, $f(y) \geq f'(y)$ for $y \in M$), then $V_f \subset V'_{f'}$.

Lemma 2.3. *Let Y be a closed set of a space M. The set of all basic neighborhoods of*

$X\times Y$ in $N(X)\times M$ forms a cofinal system of the complete neighborhood system of $X\times Y$ in $N(X)\times M$.

This lemma is obvious from Tietz extension theorem, the compactness of X and the paracompactness of M.

Let X and X' be compacta such that $\mathrm{Sh}(X)=\mathrm{Sh}(X')$. By Mardešić and Segal [15] there exist ANR-sequences $\boldsymbol{X}=\{X_k,\pi_k^{k+1}\}$ and $\boldsymbol{X}'=\{X'_k,\mu_k^{k+1}\}$ associated with X and X' respectively and maps $\boldsymbol{f}=\{f_k\}:\boldsymbol{X}\to\boldsymbol{X}'$ and $\boldsymbol{g}=\{g_k\}:\boldsymbol{X}'\to\boldsymbol{X}$ such that

$$f_k:X_{k+1}\to X'_k,\qquad g_k:X'_k\to X_k,\quad k\in J, \tag{2.4}$$

$$\mu_k^{k+1}f_{k+1}\simeq f_k\pi_{k+1}^{k+2}:X_{k+2}\to X'_k,\qquad k\in J, \tag{2.5}$$

$$\pi_k^{k+1}g_{k+1}\simeq g_k\mu_k^{k+1}:X'_{k+1}\to X_k,\qquad k\in J, \tag{2.6}$$

$$g_kf_k\simeq\pi_k^{k+1}:X_{k+1}\to X_k,\qquad k\in J, \tag{2.7}$$

$$f_kg_{k+1}\simeq\mu_k^{k+1}:X'_{k+1}\to X'_k,\qquad k\in J. \tag{2.8}$$

Let $\xi_k:X_{k+2}\times I\to X'_k$ and $\eta_k:X'_{k+1}\times I\to X_k$, $k\in J$, be homotopies realizing (2.5) and (2.6) respectively, that is, $\xi_k(x,0)=\mu_k^{k+1}f_{k+1}(x)$, $\xi_k(x,1)=f_k\pi_{k+1}^{k+2}(x)$ for $x\in X_{k+2}$ and $\eta_k(x',0)=\pi_k^{k+1}g_{k+1}(x')$, $\eta_k(x',1)=g_k\mu_k^{k+1}(x')$ for $x'\in X'_{k+1}$. Denote by $T(\boldsymbol{X})$ and $T(\boldsymbol{X}')$ the infinite telescopes associated with $\boldsymbol{X}$ and $\boldsymbol{X}'$ respectively. We define maps $F:T(\boldsymbol{X})\to T(\boldsymbol{X}')$ and $G:T(\boldsymbol{X}')\to T(\boldsymbol{X})$ as follows:

$$\begin{aligned}
&F(x,0)=f_k(x), && x\in X_{k+1},\\
&F(x,t)=\begin{cases}\xi_k(x,1-2t), & 0<t\le 1/2\\ (f_{k+1}(x),2t-1), & 1/2\le t<1\end{cases}, && (x,t)\in M_{k+1}(\boldsymbol{X}),\\
&G(x',0)=g_k(x'), && x'\in X'_k,\\
&G(x',t)=\begin{cases}\eta_k(x',1-2t), & 0<t\le 1/2\\ (g_{k+1}(x'),2t-1), & 1/2\le t<1\end{cases}, && (x',t)\in M_k(\boldsymbol{X}'),\quad k\in J.
\end{aligned}\tag{2.9}$$

($F\mid M_0(\boldsymbol{X})\cup M_1(\boldsymbol{X})$ and $G\mid M_0(\boldsymbol{X}')$ are defined as any extensions of $f_1:X_2\to X'_1$ and $g_1:X'_1\to X_1$ respectively.) From the definitions

$$F(M_{k+1}(\boldsymbol{X}))\subset M_k(\boldsymbol{X}')\qquad G(M_k(\boldsymbol{X}')\subset M_k(\boldsymbol{X}),\quad k\in J. \tag{2.10}$$

Hence F and G are proper maps. By Siebenmann [18] there exist proper homotopies $\xi:T(\boldsymbol{X})\times I\to T(\boldsymbol{X})$ and $\eta:T(\boldsymbol{X}')\times I\to T(\boldsymbol{X}')$ such that

$$\xi(x,0)=x,\qquad \xi(x,1)=GF(x),\qquad x\in T(\boldsymbol{X}), \tag{2.11}$$

$$\eta(x',0)=x'\qquad \eta(x',1)=FG(x'),\quad x'\in T(\boldsymbol{X}'), \tag{2.12}$$

$$\left.\begin{aligned}&\xi(M_{k+1}(\boldsymbol{X})\times I)\subset M_k(\boldsymbol{X})\cup M_{k+1}(\boldsymbol{X}) &(2.13)\\ &\eta(M_{k+1}(\boldsymbol{X}')\times I)\subset M_k(\boldsymbol{X}')\cup M_{k+1}(\boldsymbol{X}') &(2.14)\end{aligned}\right\},k\in J.$$

(The existence of such homotopies ξ and η is known by (2.7), (2.8) and Puppe [16, §2], too. Concrete forms of homotopies ξ and η are given in [12].)

Let Z and Z' be spaces and let N and N' be ANR's containing Z and Z' as closed sets respectively. According to Fox [6] a *mutation* $\Phi : \mathrm{U}(Z, N) \to \mathrm{U}(Z', N')$ is defined as a collection of maps $f : U \to U'$, $U \in \mathrm{U}(Z, N)$ and $U' \in \mathrm{U}(Z', N')$, such that:

$$\text{if } f : U \to U',\ f \in \Phi,\ U \supset V,\ U' \subset V',\ U, V \in \mathrm{U}(Z, N) \text{ and } U', V' \in \mathrm{U}(Z', N'), \text{ then } jfi : V \to V' \text{ belongs to } \Phi, \text{ where } i : V \to U \text{ and } j : U' \to V' \text{ are the inclusion maps,} \tag{2.15}$$

$$\text{every } U' \in \mathrm{U}(Z', N') \text{ is a range of a map } f \in \Phi, \tag{2.16}$$

$$\text{if } f_1, f_2 \in \Phi,\ f_1, f_2 : U \to U', \text{ then there exists a } V \in \mathrm{U}(Z, N) \text{ such that } V \subset U \text{ and } f_1 | V \simeq f_2 | V. \tag{2.17}$$

Two mutations $\Phi, \Psi : \mathrm{U}(Z, N) \to \mathrm{U}(Z', N')$ are *homotopic* (notation: $\Phi \simeq \Psi$) if

$$\text{for every } f \in \Phi \text{ and every } g \in \Psi \text{ such that } f, g : U \to U' \text{ there exists a } V \in \mathrm{U}(Z, N) \text{ such that } V \subset U \text{ and } f | V \simeq g | V. \tag{2.18}$$

Two spaces Z and Z' are of the *same shape* in the sense of Fox that is, $\mathrm{Sh}(Z) = \mathrm{Sh}(Z')$ if there exist ANR's N and N' containing Z and Z' as closed sets and mutations $\Phi : \mathrm{U}(Z, N) \to \mathrm{U}(Z', N')$ and $\Psi : \mathrm{U}(Z', N') \to \mathrm{U}(Z, N)$ such that

$$\Psi\Phi \simeq 1_Z \qquad \Phi\Psi \simeq 1_{Z'}. \tag{2.19}$$

Here 1_Z is the mutation generated by the identity: $Z \to Z$.

3. Shape of product spaces

Theorem 3.1. *Let X, X' be compacta and let Y, Y' be metrizable spaces. If* $\mathrm{Sh}(X) = \mathrm{Sh}(X')$ *and* $\mathrm{Sh}(Y) = \mathrm{Sh}(Y')$, *then* $\mathrm{Sh}(X \times Y) = \mathrm{Sh}(X' \times Y')$.

Proof. It is enough to give the proof in each of the following two cases: (1) $X = X'$ and (2) $Y = Y'$. The proof for the case (1) is simple. Let M and M' be ANR's containing Y and Y' as closed sets respectively. Since $\mathrm{Sh}(Y) = \mathrm{Sh}(Y')$ there exist mutations $\Phi : \mathrm{U}(Y, M) \to \mathrm{U}(Y', M')$ and $\Psi : \mathrm{U}(Y', M') \to \mathrm{U}(Y, M)$ such that

$$\Psi\Phi \simeq 1_Y \quad \text{and} \quad \Phi\Psi \simeq 1_{Y'}. \tag{3.1}$$

(See Section 2 for notations.) Imbed X into an arbitrary ANR N. For each map $f : U \to U'$, $f \in \Phi$, $U \in \mathrm{U}(Y, M)$ and $U' \in \mathrm{U}(Y', M')$, consider the map $\tilde{f} = 1_N \times f : N \times U \to N \times U'$, where 1_N is the identity: $N \to N$. By the compactness of X, the set consisting of all maps of the form $\tilde{f}$ and their restrictions to neighborhoods of $X \times Y$ in $N \times M$ forms a mutation $\tilde{\Phi} : \mathrm{U}(X \times Y, N \times M) \to \mathrm{U}(X \times Y', N \times M')$. Similarly the set of maps of the form $\tilde{g} =$

$1_N \times g : N \times U' \to N \times U$, where $g : U' \to U$, $g \in \Psi$, $U' \in \mathrm{U}(Y', M')$ and $U \in \mathrm{U}(Y, M)$, and their restrictions to neighborhoods of $X \times Y'$ in $N \times M'$ forms a mutation $\bar{\Psi} : \mathrm{U}(X \times Y', N \times M') \to \mathrm{U}(X \times Y, N \times M)$. By (3.1) we have $\bar{\Psi}\bar{\Phi} \simeq 1_{N \times M}$ and $\bar{\Phi}\bar{\Psi} \simeq 1_{N \times M'}$. This implies $\mathrm{Sh}(X \times Y) = \mathrm{Sh}(X \times Y')$.

Next we consider the case (2). Let M be an ANR containing Y as a closed set. Choose ANR-sequences $\boldsymbol{X} = \{X_k, \pi_k^{k+1}\}$ and $\boldsymbol{X}' = \{X'_k, \mu_k^{k+1}\}$ associated with X and X' satisfying the conditions (2.4)–(2.8) and construct the AR's $N(\boldsymbol{X})$ and $N(\boldsymbol{X}')$ associated with $\boldsymbol{X}$ and $\boldsymbol{X}'$. There exist proper maps $F : T(\boldsymbol{X}) \to T(\boldsymbol{X}')$, $G : T(\boldsymbol{X}') \to T(\boldsymbol{X})$, proper homotopies $\xi : T(\boldsymbol{X}) \times I \to T(\boldsymbol{X})$ and $\eta : T(\boldsymbol{X}') \to T(\boldsymbol{X})$ satisfying the conditions (2.11)–(2.14), where $T(\boldsymbol{X})$ $(= N(\boldsymbol{X}) - X)$ and $T(\boldsymbol{X}')$ $(= N(\boldsymbol{X}') - X')$ are the infinite telescopes associated with $\boldsymbol{X}$ and $\boldsymbol{X}'$ respectively. (See Section 2 for notations.) For each map $f \in C(M, R_0)$, define $\Phi_f : N(\boldsymbol{X}) \times M \to N(\boldsymbol{X}') \times M$ by

$$\Phi_f(x, y) = (F \times 1_M)\alpha_f(x, y) \quad \text{for } (x, y) \in N(\boldsymbol{X}) \times M. \tag{3.2}$$

Here α_f is the map of $N(\boldsymbol{X}) \times M$ into itself defined for f as in (2.2). Since $\alpha_f(N(\boldsymbol{X}) \times M) = M_f \subset T(\boldsymbol{X})$, Φ_f is well-defined. Let Φ' be the set consisting of all maps of the form Φ_f, $f \in C(M, R_0)$. Define the set $\boldsymbol{\Phi}$ consisting of restrictions of maps in Φ' as follows:

$\varphi \in \boldsymbol{\Phi}$ if and only if there exist $f, g \in C(M, R_0)$, $f > g$, $W \in \mathrm{U}(X \times Y, N(\boldsymbol{X}) \times M)$, $W' \in \mathrm{U}(X' \times Y, N(\boldsymbol{X}') \times M)$ and $V \in \mathrm{U}(Y, M)$ such that $\varphi = \Phi_f \mid W : W \to W'$ and $W \supset V_g = N(\boldsymbol{X}) \times V - M_g$, where M_g is the g-set of $N(\boldsymbol{X}) \times M$ defined for g as in (2.1). (3.3)

We shall show that $\boldsymbol{\Phi}$ is a mutation of $\mathrm{U}(X \times Y, N(\boldsymbol{X}) \times M)$ into $\mathrm{U}(X' \times Y, N(\boldsymbol{X}') \times M)$. Since $\boldsymbol{\Phi}$ satisfies obviously the condition (2.15), it is enough to prove that (2.16) and (2.17) are satisfied. Let $W' \in \mathrm{U}(X' \times Y, N(\boldsymbol{X}') \times M)$. Since $(F \times 1_M)^{-1} W' \in \mathrm{U}(X \times Y, N(\boldsymbol{X}) \times M)$, by Lemma 2.3 there exists a basic neighborhood $W = N(\boldsymbol{X}) \times V - M_g$, $V \in \mathrm{U}(Y, M)$, $g \in C(M, R_0)$, such that $W \subset (F \times 1_M)^{-1} W'$. Define $f \in C(M, R_0)$ by $f(y) = g(y) + 1$, $y \in M$. Since $f > g$, we have $\alpha_f(W) \subset W$ and hence $\Phi_f(W) \subset W'$. Thus the map $\Phi_f \mid W : W \to W'$ belongs to $\boldsymbol{\Phi}$. To prove that (2.17) is satisfied, let $\varphi, \varphi' : W \to W'$ be maps in $\boldsymbol{\Phi}$, $W \in \mathrm{U}(X \times Y, N(\boldsymbol{X}) \times M)$ and $W' \in \mathrm{U}(X' \times Y, N(\boldsymbol{X}') \times M)$. By (3.3) there exist $f, f', g, g' \in C(M, R_0)$, $f > g$, $f' > g'$, $V, V' \in \mathrm{U}(Y, M)$ such that $\varphi = \Phi_f \mid W$, $\varphi' = \Phi_{f'} \mid W$, $W \supset N(\boldsymbol{X}) \times V - M_g$ and $W \supset N(\boldsymbol{X}) \times V' - M_{g'}$. Put $V'' = V \cap V'$, $h = \min(g, g')$ and $W'' = N(\boldsymbol{X}) \times V'' - M_h$. Then $W \supset W'' \in \mathrm{U}(X \times Y, N(\boldsymbol{X}) \times M)$. By Lemma 2.2 $\varphi \mid W'' \simeq \varphi' \mid W''$. Thus $\boldsymbol{\Phi}$ is a mutation. Similarly we define a mutation $\boldsymbol{\Psi} : \mathrm{U}(X' \times Y, N(\boldsymbol{X}') \times M) \to \mathrm{U}(X \times Y, N(\boldsymbol{X}) \times M)$ as follows. For each map $f \in C(M, R_0)$, set

$$\Psi_f = (G \times 1_M)\alpha'_f : N(\boldsymbol{X}') \times M \to N(\boldsymbol{X}) \times M. \tag{3.4}$$

Here α'_f is the map of $N(\boldsymbol{X}') \times M$ into itself defined for f as in (2.2). Let Ψ' be the set consisting of all maps of the form Ψ_f, $f \in C(M, R_0)$. A mutation $\mathbf{\Psi}: \mathrm{U}(X' \times Y, N(X') \times M) \to \mathrm{U}(X \times Y, N(X) \times M)$ is defined as follows.

$$\begin{array}{l} \psi \in \mathbf{\Psi} \text{ if and only if there exists } f, g \in C(M, R_0),\ f > g, \\ W' \in \mathrm{U}(X' \times Y,\ N(X') \times M),\ W \in \mathrm{U}(X \times Y,\ N(X) \times M) \text{ and} \\ V \in \mathrm{U}(Y, M) \text{ such that } \psi = \Psi_f \mid W' : W' \to W, \\ W' \supset N(X') \times V - M'_g, \text{ where } M'_g \text{ is the } g\text{-set of } N(X') \times M \\ \text{defined by } g \text{ as in (2.1).} \end{array} \tag{3.5}$$

To complete the proof it remains to prove:

$$\mathbf{\Psi\Phi} \simeq 1_{X \times Y}, \tag{3.6}$$

$$\mathbf{\Phi\Psi} \simeq 1_{X' \times Y}. \tag{3.7}$$

We shall prove only the relation (3.6). The proof for (3.7) is similarly given and we omit it. Let $\varphi : W \to W'$, $\varphi \in \mathbf{\Phi}$, and $\psi : W' \to W''$, $\psi \in \mathbf{\Psi}$, where $W, W'' \in \mathrm{U}(X \times Y, N(X) \times M)$ and $W' \in \mathrm{U}(X' \times Y, N(X') \times M)$. By (3.3) and (3.5) there exist $V, V' \in \mathrm{U}(Y, M)$, $f > g$ and $f' > g'$, $f, g, f', g' \in C(M, R_0)$, such that $\varphi = \Phi_f \mid W$, $\psi = \Psi_{f'} \mid W'$, $W \supset N(X) \times V - M_g$ and $W' \supset N(X') \times V' - M_{g'}$. Let $V'' = V \cap V'$. By the properties (2.10) and (2.13) of the maps F, G and the homotopy ξ, there exists $h \in C(M, R_0)$ such that if we put $V''_h = N(X) \times V'' - M_h$ we have

$$\alpha_h \mid V''_h \simeq (GF \times 1_M)\alpha_h \mid V''_h \quad \text{in } W. \tag{3.8}$$

Let $h' \in C(M, R_0)$ such that $h' > \max(h, f, f')$. Then, by (2.3) $\alpha'_{h'}$ is the identity on the set $(F \times 1_M)\alpha_h(N(X) \times M)$. Hence

$$(G \times 1_M)\alpha'_{h'}(F \times 1_M)\alpha_h \mid V''_h = (GF \times 1_M)\alpha_h \mid V''_h. \tag{3.9}$$

By Lemma 2.2, (3.8) and (3.9) we have

$$\begin{aligned} \psi\varphi \mid V''_h = (G \times 1_M)\alpha'_{f'}(F \times 1_M)\alpha_f \mid V''_h &\simeq (G \times 1_M)\alpha'_{h'}(F \times 1_M)\alpha_h \mid V''_h \\ \simeq \alpha_h \mid V''_h \simeq 1_{V_h}. \quad &\text{in } W''. \end{aligned}$$

Thus the relation (3.7) is proved. This completes the proof of Theorem 3.1.

Remark 3.2. In the shape category consisting of metrizable spaces and mutations or generally topological spaces and shapings (cf. Mardešić [13]), the existence of products is not known. Keesling [9] proved that there exists a separable metric space X such that $\mathrm{Sh}(X \times X)$ is not the product of $\mathrm{Sh}(X)$ and $\mathrm{Sh}(X)$. For a compactum X and a metrizable space Y, it is interesting to know whether $\mathrm{Sh}(X \times Y)$ is the product of $\mathrm{Sh}(X)$ and $\mathrm{Sh}(Y)$ or not.

Remark 3.3. Godlewski and Nowak [8] gave the following example: X is a Warsaw circle, X' is a 1-sphere and Y is a countable discrete space. Then $\mathrm{Sh}_s(X \times Y) \neq \mathrm{Sh}_s(X' \times Y)$, where $\mathrm{Sh}_s(Z)$ is the strong shape of Z in the sense of Borsuk ([2] or [3, Ch. III]). Therefore it is known that we can not replace in Theorem 3.1 the shape

Sh in the sense of Fox by the strong shape Sh_s in the sense of Borsuk. In this example, it is obvious that $\mathrm{Sh}(X \times Y)$ is the product of $\mathrm{Sh}(X)$ and $\mathrm{Sh}(Y)$.

Remark 3.4. Mardešić [15] defined shape for general topological spaces and proved that it is equal to the shape in the sense of Fox for metrizable spaces. By modifying the proof we can obtain the following generalization of Theorem 3.1.

Theorem 3.5. *If X, X' are compacta, Y, Y' are paracompact spaces such that $\mathrm{Sh}(X) = \mathrm{Sh}(X')$ and $\mathrm{Sh}(Y) = \mathrm{Sh}(Y')$ then $\mathrm{Sh}(X \times Y) = \mathrm{Sh}(X' \times Y')$, where $\mathrm{Sh}(Z)$ is the shape of Z in the sense of Mardešić.*

The *fundamental dimension* $\mathrm{Fd}(X)$ of a space X is defined by Borsuk [3, Ch. VII] as follows:

$$\mathrm{Fd}(X) = \mathrm{Min}\{\dim X' : \mathrm{Sh}(X) \leq \mathrm{Sh}(X')\}.$$

As a consequence of Theorem 3.1, we prove the following product theorem for fundamental dimension.

Corollary 3.6. *If X is a compactum and Y is a space, then $\mathrm{Fd}(X \times Y) \leq \mathrm{Fd}(X) + \mathrm{Fd}(Y)$.*

Proof. First we note that in the first part of the proof of Theorem 3.1 the following fact is proved.

If X is a compactum, Y and Y' are spaces such that $\mathrm{Sh}(Y) \leq \mathrm{Sh}(Y')$ then $\mathrm{Sh}(X \times Y) \leq \mathrm{Sh}(X \times Y')$. (3.10)

By [10, Theorem 3], there exists a Δ-space X' such that $\mathrm{Sh}(X) = \mathrm{Sh}(X')$ and $\mathrm{Fd}(X) = \dim X'$. If $\mathrm{Sh}(Y) \leq \mathrm{Sh}(Y')$ then $\mathrm{Sh}(X \times Y) = \mathrm{Sh}(X' \times Y) \leq \mathrm{Sh}(X' \times Y')$. Here the first equality follows from Theorem 3.1 and the second relation follows from (3.10). Since $\dim(X' \times Y') = \dim X' + \dim Y'$, the corollary is obtained.

Corollary 3.7. *If X is an FAR (see Borsuk [3, Ch. VIII]) and Y is a space, then the projection: $X \times Y \to Y$ induces a shape equivalence. In particular $\mathrm{Sh}(X \times Y) = \mathrm{Sh}(Y)$.*

Proof. Let P be a space consisting of one point. Then $\mathrm{Sh}(X) = \mathrm{Sh}(P)$ by [3, Ch. VI (6.6)]. If we denote by f a unique map of X into P, then from the proof of Theorem 3.1 it follows that the map $f \times 1_Y : X \times Y \to P \times Y$ induces a shape equivalence.

Theorem 3.8. *Let X be an FANR (resp. FAR) and Y an MANR (resp. MAR). Then $X \times Y$ is an MANR (resp. MAR).*

For the definition of MANR and MAR, see [7]. For the proof we need the following lemma.

Lemma 3.9. *A compactum X is an* FANR *(resp.* FAR*) if and only if for every compact* AR *M containing X there exist a decreasing sequence $\{W_k : k \in J\}$ (resp. $\{W_k : k \in J\}$, $W_1 = M$) of closed neighborhoods of X in M and a map $\varphi : M \times R_0 \to M$ satisfying the following conditions:*

$$\{W_k\} \text{ forms a neighborhood basis.} \tag{3.11}$$

$$\varphi(x, 0) = x \quad \text{for } x \in M. \tag{3.12}$$

$$\varphi(W_k \times [k+n, \infty)) \subset W_{k+n-1} \quad \text{for } k, n \in J. \tag{3.13}$$

$$\varphi(x, r) = x \quad \text{for } x \in W_{k+2} \text{ and } 0 \leq r \leq k, \quad k \in J. \tag{3.14}$$

In the case of FANR this is [5, Lemma 4]. The proof is easily obtained by making use of [17, Theorem 5.8]. For FAR, since every FAR is of trivial shape, the lemma is easily proved by Chapman [4, Theorem 2].

Proof of Theorem 3.8. We give only the proof in the case of FANR. Let M be a compact AR containing X. By Lemma 3.8 there exist a decreasing sequence $\{W_k : k \in J\}$ of closed neighborhoods of X in M and a map $\varphi : M \times R_0 \to M$ satisfying (3.11)–(3.14). Let N be an ANR containing Y as a closed set. Since Y is an FANR, there exist a closed neighborhood Y' of Y in N and a mutational retraction $\Psi : \mathrm{U}(Y', N) \to \mathrm{U}(Y, N)$. Denote by $C(N, R_0)$ the set of all maps of N into R_0. Let $f \in C(N, R_0)$ and $\psi \in \Psi$, $\psi : V' \to V$, $V' \in \mathrm{U}(Y', N)$, $V \in \mathrm{U}(Y, N)$. Define a map $\Phi(f, \psi) : M \times V' \to M \times V$ by $\Phi(f, \psi)(x, y) = (\varphi(x, f(y)), \psi(y))$, $(x, y) \in M \times V'$. Consider the set Φ' of maps of the form $\Phi(f, \psi)$, $f \in C(N, R_0)$ and $\psi \in \Psi$. By the same way as in the proof of Theorem 3.1, it is proved that Φ' generates a mutation $\Phi : \mathrm{U}(W_1 \times Y', M \times N) \to \mathrm{U}(X \times Y, M \times N)$. This is showed by making use of the properties (3.11)–(3.13) of the map φ and $\{W_k\}$. Moreover, since $\varphi(x, r) = x$ for $x \in X$ and $r \in R_0$ by (3.14) and $\psi(y) = y$ for $y \in Y$, $\Phi(f, \psi)(x, y) = (x, y)$ for $(x, y) \in X \times Y$. Hence Φ is a mutational retraction of $\mathrm{U}(W_1 \times Y', M \times N)$ into $\mathrm{U}(X \times Y, M \times N)$. Therefore $X \times Y$ is a neighborhood mutational retract of $M \times N$. This completes the proof.

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